

# MODULAR ADJACENCY ALGEBRAS OF GRASSMANN GRAPHS

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**ABSTRACT.** The adjacency algebra of an association scheme is defined over an arbitrary field. In general, it is always semisimple over a field of characteristic zero but not always semisimple over a field of positive characteristic. The structures of adjacency algebras over fields of positive characteristic have not been sufficiently studied.

In this paper, we consider the structures of adjacency algebras of some  $P$ -polynomial schemes of class  $d$  with intersection numbers  $c_i \not\equiv 0$  modulo  $p$  for  $1 \leq i \leq d$  over fields of positive characteristic  $p$ . The classes of these  $P$ -polynomial schemes include association schemes originating from Grassmann graphs, double Grassmann graphs, and all types of dual polar graphs. We discuss the structures of the modular adjacency algebras of Grassmann graphs.

## 1. INTRODUCTION

An adjacency algebra of an association scheme is defined over an arbitrary field. In general, it is always semisimple over a field of characteristic zero but not always semisimple over a field of positive characteristic. An adjacency algebra of an association scheme over a field of positive characteristic is called a modular adjacency algebra. Hanaki and the second author of this paper determined the structure of the modular adjacency algebras and the modular standard modules of association schemes of class 2 [6]. Using modular standard modules, they provided more detailed classification than using parameters of strongly regular graphs. This indicates that structures of the modular standard modules of association schemes provide more detailed characterization than parameters of association schemes. In order to determine the structure of the modular standard modules, we first need to obtain the structure of the modular adjacency algebras. However, the structure of modular adjacency algebras has not been sufficiently studied (see [9], [11], [12]).

In this paper, we will consider the structure of modular adjacency algebras of  $P$ -polynomial schemes with the intersection numbers  $c_i \not\equiv 0$  modulo  $p$  for  $1 \leq i \leq d$ . The class of these  $P$ -polynomial schemes includes association schemes originating from Grassmann graphs, double Grassmann graphs, and all types of dual polar graphs. These schemes have an additional condition for the intersection numbers:  $b_i \equiv 0$  for  $1 \leq i \leq d-1$ . In particular, we will discuss the structure of the modular adjacency algebras of Grassmann graphs and determine the structure of the modular adjacency algebras of Grassmann graphs with some special parameters. Although Grassmann graphs are  $q$ -analogues of Johnson graphs, we obtain the correspondence of the structure of the modular adjacency algebras of Grassmann graphs to the structure of the modular adjacency algebras of Johnson graphs.

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2000 *Mathematics Subject Classification.* Primary 05E30.

*Key words and phrases.* association schemes;  $P$ -polynomial schemes; modular adjacency algebras; Grassmann graphs; dual polar graphs.

## 2. PREPARATION

**2.1. Association schemes.** Let  $X$  be a finite set with cardinality  $n$ . We define  $R_0 := \{(x, x) \mid x \in X\}$ . Let  $R_i \subseteq X \times X$  be given. We set  $R_{i^*} := \{(z, y) \mid (y, z) \in R_i\}$ . Let  $S$  be a partition of  $X \times X$  such that  $R_0 \in S$  and the empty set  $\emptyset \notin S$ , and we assume that  $R_{i^*} \in S$  for each  $R_i \in S$ . Then, the pair  $\mathfrak{X} = (X, S)$  will be called an association scheme if, for all  $R_i, R_j, R_k \in S$ , there exists a cardinal number  $p_{ij}^k$  such that, for all  $(y, z) \in R_k$ ,

$$\#\{x \in X \mid (y, x) \in R_i, (x, z) \in R_j\} = p_{ij}^k.$$

The numbers  $p_{ij}^k$  are called the intersection numbers of  $\mathfrak{X}$ . The valency of  $R_i$  is the intersection number  $p_{ii^*}^0$ , and it is denoted by  $n_i$ .

For each  $R_i \in S$ , we define the  $n \times n$  matrix  $A_i$  indexed by the elements of  $X$ , as

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

We call  $A_i$  the adjacency matrix of  $R_i$ .

Let the cardinal number of  $S$  be  $d + 1$ , and let  $J$  be an  $n \times n$  matrix in which each element is unity. Then, by definition, it follows that  $\sum_{i=0}^d A_i = J$ . We have that, for all  $A_i, A_j$ ,

$$A_i A_j = \sum_{k=0}^d p_{ij}^k A_k.$$

The set of linear combination of all adjacency matrices is closed under multiplication, because of the above equation. For a commutative ring  $R$  with identity, we define  $R\mathfrak{X} = \bigoplus_{i=0}^d R A_i$  as a matrix ring over  $R$ ; this is called an adjacency algebra of  $\mathfrak{X}$  over  $R$ . A modular adjacency algebra is an adjacency algebra over a field of positive characteristic  $p$ . The association scheme is called symmetric if  $R_i = R_{i^*}$  for all  $R_i \in S$ . The association scheme is called commutative if  $p_{ij}^k = p_{ji}^k$  for any  $R_i, R_j$  and  $R_k \in S$ . The adjacency algebra of a symmetric association scheme is commutative.

We define the  $i$ -th intersection matrix  $B_i$  by  $(B_i)_{jk} = p_{ij}^k$  and  $RB = \bigoplus_{i=0}^d R B_i$  as a matrix ring over  $R$ ; this is called the intersection algebra of  $\mathfrak{X}$ . Then the map  $\varphi : R\mathfrak{X} \rightarrow RB$  given by  $\varphi(A_i) = B_i$ , is an algebra isomorphism.

**2.2. Distance-regular graphs and  $P$ -polynomial schemes.** Let  $\Gamma$  be a connected undirected graph on  $X$ . A path of length  $r$  from  $x$  to  $y$  ( $x, y \in X$ ) is a sequence of vertices  $x_0 = x, x_1, \dots, x_{r-1}, x_r = y$  such that each  $(x_i, x_{i+1})$  is an edge of  $\Gamma$ . The distance of  $x$  and  $y$  is the minimum length of paths from  $x$  to  $y$  and is denoted by  $\delta(x, y)$ ;  $\delta(x, x)$  is defined to be zero. We can easily check that  $\delta(x, y)$  satisfies the axioms of distance. The diameter of  $\Gamma$  is the maximum distance between two vertices and is denoted by  $d$ . Let  $R_i$  be the relation on  $X$  defined by  $(x, y) \in R_i$  if and only if  $\delta(x, y) = i$ . If  $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$  becomes an association scheme,  $\Gamma$  is called distance-regular and, obviously,  $\mathfrak{X}$  is always symmetric.

A symmetric association scheme  $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$  is called a  $P$ -polynomial scheme with respect to the ordering  $R_0, R_1, \dots, R_d$  if there exist complex coefficient polynomials  $v_i(x)$  of degree  $i$  ( $0 \leq i \leq d$ ) such that  $A_i = v_i(A_1)$ , where  $A_i$  is the adjacency matrix with respect to  $R_i$ .

When  $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$  is a  $P$ -polynomial scheme, there are nonnegative integers  $a_i, b_i, c_i$  such that

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (0 \leq i \leq d-1),$$

where we define  $b_{-1} = 0$  and  $A_{-1} = O$ . Then we have  $b_0 = n_1$ ,  $a_0 = 0$ , and  $c_1 = 1$ . It is known that  $a_i + b_i + c_i = b_0$  ( $0 \leq i \leq d$ ), where we define  $b_d = 0$  and  $c_0 = 0$ . We define the intersection array by  $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$ .

It is known for valencies that

$$n_0 = 1, n_1 = b_0, n_{i+1} = n_i \frac{b_i}{c_{i+1}}.$$

Let  $\Gamma$  be a distance-regular graph, let  $X$  be the vertex set of  $\Gamma$ , and let  $d$  be the maximum distance of  $\Gamma$ . We define the relation on  $X$  by  $(x, y) \in R_i$  if and only if  $\delta(x, y) = i$  for  $x, y \in X$ . Then  $(X, \{R_i\}_{0 \leq i \leq d})$  is a  $P$ -polynomial scheme.

### 3. GRASSMANN GRAPHS

Let  $V$  be an  $m$ -dimensional vector space over a finite field with  $q$  elements, and let  $\binom{V}{d}$  be the set of all  $d$ -dimensional subspaces of  $V$  ( $2d \leq v$ ). For each  $0 \leq i \leq d$ , we define:

$$R_i := \left\{ (N_1, N_2) \in \binom{V}{d} \times \binom{V}{d} \mid \dim(N_1 \cap N_2) = d - i \right\}.$$

Then the relations  $R_0, R_1, \dots, R_d$  form an association scheme on the set  $\binom{V}{d}$  with  $d$  classes; this is called the Grassmann scheme or the  $q$ -analogue Johnson scheme  $J_q(v, d)$ . The Grassmann scheme  $J_q(v, d)$  is accompanied by a regular semilattice [5]. We can compute the structure constants using the parameters of these partially ordered sets (posets). Let  $\{C_i\}_{0 \leq i \leq d}$  be a basis with

$$(1) \quad C_i = \sum_{u=0}^d \binom{u}{i}_q A_{d-u} \quad (0 \leq i \leq d),$$

where  $\binom{a}{b}_q$  is a Gaussian coefficient. We consider homomorphisms between modular adjacency algebras  $FJ_q(v, d)$  and  $FJ_q(v-2, d-1)$  using structure constants of  $\{C_i\}_{0 \leq i \leq d}$ . The structure constants  $\{\rho_{r,s}^t\}_{0 \leq s, t, r \leq d}$  satisfy the following;

$$C_r C_s = \sum_{t=0}^{\min(r,s)} \rho_{r,s}^t C_t.$$

We can calculate the structure constants as follows.

**Lemma 1.**

$$\rho_{r,s}^t = \binom{d-t}{r-t}_q \binom{d-t}{s-t}_q \binom{v-s-r}{v-d-t}_q q^{t(d-s-r+t)}$$

for all  $r, s, t = 0, 1, \dots, d$ .

**Notation 1.** As  $q \rightarrow 1$ , the Gaussian coefficient becomes the usual binomial coefficient. Therefore, the structure constants of  $J_q(v, d)$  coincide with those of the Johnson scheme  $J(v, d)$  [4].

To prove Lemma 1, we use the following properties of the Gaussian coefficients; see Proposition 8.5.2 in [3].

**Proposition 2.**

(i) For  $n \in \mathbb{R}$  and  $m, k \in \mathbb{Z}$

$$\binom{m}{n}_q \binom{n}{k}_q = \binom{m}{k}_q \binom{m-k}{n-k}_q;$$

(ii) for  $l$  a nonnegative integer,

$$\begin{aligned} \sum_{k+m=0}^{k+m=l} (-1)^k \binom{l}{m+k}_q \binom{s+k}{n}_q q^{\binom{k}{2}+k(m-n)} \\ = (-1)^{l+m} q^{(l-m)(l-2n+m-1)/2} \binom{s-m}{n-l}_q; \end{aligned}$$

(iii)

$$\begin{aligned} \sum_{k+m=0}^{k+m=l} (-1)^k \binom{l}{m+k}_q \binom{t-k}{n}_q q^{\binom{k}{2}+km} \\ = (-1)^m q^{-\binom{m}{2}+l(t+m-n)} \binom{t-l+m}{n-l}_q. \end{aligned}$$

*Proof of Lemma 1.* We use the same parameters of regular semilattices that we used in [5], and we note that the following equations are well known [3, 5]:

$$(2) \quad \rho_{r,s}^t = \mu(r, t) \left[ \sum_{j=0}^t \nu'(j, t) \pi(j, r, s) \right],$$

where

$$(3) \quad \begin{aligned} \mu(t, r) &= \binom{n-t}{n-r}_q = \binom{n-t}{r-t}_q \quad \text{for } t \leq r, \\ \nu'(j, t) &= (-1)^{t-j} q^{\binom{t-j}{2}} \binom{t}{j}_q, \end{aligned}$$

and

$$\pi(j, r, s) = \sum_{i=\max\{0, r+s-n\}}^{\min\{j, s\}} q^{(j-i)(s-i)} \binom{j}{i}_q \binom{n-j}{s-i}_q \binom{m-r-s+i}{m-n}_q \quad \text{for } r-j \leq n-s, \ j \leq r.$$

The summation of (2) implies the following equation:

$$\begin{aligned} \sum_{j=0}^t \nu'(j, t) \pi(j, r, s) \\ = \sum_{j=0}^t \sum_{i=\max\{0, r+s-n\}}^{\min\{j, s\}} (-1)^{t-j} q^{\binom{t-j}{2}} \binom{t}{j}_q q^{(j-i)(s-i)} \binom{j}{i}_q \binom{n-j}{s-i}_q \binom{m-r-s+i}{m-n}_q. \end{aligned}$$

Since  $(j-i)(s-i) = (t-i)(s-i) + (t-j)(i-s)$  and using (i) in Proposition 2,

$$(4) \quad \begin{aligned} \sum_{j=0}^t \nu'(j, t) \pi(j, r, s) &= \sum_{i=\max\{0, r+s-n\}}^t q^{(t-i)(s-i)} \binom{t}{i}_q \binom{m-r-s+i}{m-n}_q \\ &\quad \times \sum_{j=i}^t (-1)^{t-j} \binom{t-i}{t-j}_q \binom{n-j}{s-i}_q q^{(t-j)(i-s) + \binom{t-j}{2}}. \end{aligned}$$

Hence, using (ii) in Proposition 2, we can transform the second summation of (4) to the following form:

$$(-1)^{t-i} q^{\binom{t-i}{2} - (t-i)(s-i)} \binom{n-t}{s-i-t+i}_q.$$

Then we have

$$(5) \quad \sum_{j=0}^t \nu'(j, t) \pi(j, r, s) = \sum_{i=\max\{0, r+s-n\}}^t (-1)^{t-i} \binom{t}{t-i}_q \binom{m-s-(r-i)}{m-n}_q q^{\binom{t-i}{2}} \binom{n-t}{s-t}_q.$$

Also, by using (iii) in Proposition 2, we have

$$(6) \quad \sum_{j=0}^t \nu'(j, t) \pi(j, r, s) = q^{t(n-s-r+t)} \binom{m-s-r}{m-n-t}_q \binom{n-t}{s-t}_q.$$

Then (2), (3), and (6) imply Lemma 1.  $\square$

**Theorem 3** ([9]). *Let  $F$  be a field of characteristic  $p$ , and let  $q$  be a prime power that is not necessarily related to  $p$ . Then*

$$\varphi_d : FJ_q(v, d) \rightarrow FJ_q(v-2, d-1) \quad (C_i^d \mapsto C_{i-1}^{d-1})$$

*is an algebraic epimorphism, where  $C_i^d$  (or  $C_i^{d-1}$ ) is an  $F$ -basis of  $FJ_q(v, d)$  (or  $FJ_q(v-2, d-1)$ ) as per the above definition (1).*

**3.1. Structures of Grassmann graphs.** We use the following properties of the Gaussian coefficient. The following lemma is proved by a combinatorial interpretation of the Gaussian coefficients as a polynomial in  $q$  [10].

**Lemma 4.** *Let  $p$  be a prime, let  $q_1, q_2$  be prime powers such that  $q_1 \equiv q_2 \pmod{p}$ , and let  $v, d$  be positive integers. Then*

$$\binom{v}{d}_{q_1} \equiv \binom{v}{d}_{q_2} \pmod{p}.$$

*In particular, if  $q \equiv 0 \pmod{p}$ , then*

$$\binom{v}{d}_q \equiv 1 \pmod{p},$$

*and if  $q \equiv 1 \pmod{p}$ , then*

$$\binom{v}{d}_q \equiv \binom{v}{d} \pmod{p}.$$

By Lemma 4, we can prove the following isomorphisms.

**Theorem 5.** *If  $q$  is a prime power of  $p$ , then*

$$FJ_q(v, d) \cong FJ_p(v, d).$$

*If  $q_1, q_2$  be prime powers such that  $q_1 \equiv q_2 \pmod{p}$ , then*

$$FJ_{q_1}(v, d) \cong FJ_{q_2}(v, d).$$

*In particular, if  $q \equiv 1 \pmod{p}$ , then*

$$FJ_q(v, d) \cong FJ(v, d).$$

The correspondence between the basis of  $FJ_q(v, d)$  and that of  $FJ_p(v, d)$  is  $C_i^d \mapsto C_i^d$ . In latter cases, it may seem too difficult to describe these algebras as direct sums of factor algebras of polynomial rings. When  $q$  is equivalent to one,  $FJ_q(v, d)$  is isomorphic to a modular adjacency algebra of the Johnson scheme  $FJ(v, d)$ . We studied the structure of  $FJ(v, d)$  in [9] and determined that it is complicated.

We consider the decomposition of  $FJ_p(v, d)$  to indecomposable two-sided ideals (*blocks*) [7]. We denote the number of blocks of  $FJ_p(v, d)$  by  $k(FJ_p(v, d))$ . Since this number is calculated as follows [8]:

$$k(FJ_p(v, d)) = \left\{ i \in \{0, \dots, d\} \mid p \nmid \binom{v-2i}{v-d-i}_p p^{i(d-i)} \right\},$$

the number of blocks of  $FJ_p(v, d)$  is equal to two and does not depend on  $v$  since the index of  $p$  determines whether  $p$  divides  $\binom{v-2i}{v-d-i}_p p^{i(d-i)}$ . Actually, principal blocks of algebras are  $F$  because  $p$  does not divide  $\binom{v}{d}_q$ ; this is proved by Proposition 4 in [6].

**Theorem 6.** *For any  $v, v' (\geq 2d)$ ,*

$$FJ_p(v, d) \cong FJ_p(v', d) \cong F \oplus D$$

*where  $F$  is a principal block.*

We now have that the structure of  $FJ_q(v, d)$  is isomorphic to an algebra that does not depend on  $v$  when  $q$  is a power  $p$ . In the next section, we will describe this algebra as a direct sum of factor algebras of polynomial rings.

#### 4. THE MODULAR ADJACENCY ALGEBRA OF A $P$ -POLYNOMIAL SCHEME WITH $c_i \not\equiv 0$ FOR $1 \leq i \leq d$

We fix a prime number  $p$ . Let  $F$  be a field of characteristic  $p$ , and let  $F\mathfrak{X}$  be the adjacency algebra over the field  $F$  of a  $P$ -polynomial scheme  $\mathfrak{X}$  of class  $d$  with the intersection numbers  $c_i \not\equiv 0$  for  $1 \leq i \leq d$ . We assume that the field  $F$  is the splitting field of  $F\mathfrak{X}$ .

From our assumption, we have

$$B_1 = \begin{pmatrix} 0 & 1 & & & & \\ b_0 & a_1 & c_2 & & & 0 \\ & b_1 & a_2 & \ddots & & \\ & & b_2 & \ddots & & \\ & & & \ddots & & c_d \\ 0 & & & & b_{d-1} & a_d \end{pmatrix}.$$

Since we assume that  $c_i \not\equiv 0 \pmod{p}$  for  $1 \leq i \leq d$ , the adjacency algebra  $F\mathfrak{X}$  is generated only by the element  $A_1$ . We define a Jordan block  $J(s, \lambda)$  by the following  $s \times s$  matrix:

$$J(s, \lambda) = \begin{pmatrix} \lambda & 1 & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & 1 \\ 0 & & & 0 & \lambda \end{pmatrix}.$$

Then the following theorem holds.

**Theorem 7.** *We fix a prime number  $p$ . Let  $\mathfrak{X}$  be a  $P$ -polynomial scheme of class  $d$  with the intersection numbers  $c_i \not\equiv 0$  for  $1 \leq i \leq d$ , and let  $F$  be a field of characteristic  $p$  such that the field  $F$  is the splitting field of the adjacency algebra  $F\mathfrak{X}$ .*

*We assume that the intersection matrix  $B_1$  is similar to the Jordan normal form  $J(s_1, \lambda_1) \oplus J(s_2, \lambda_2) \oplus \dots \oplus J(s_l, \lambda_l)$  over the field  $F$ .*

Then, we have

$$F\mathfrak{X} \cong \bigoplus_{i=1}^l F[x_i]/(x_i^{s_i}).$$

Moreover, it follows that  $\lambda_i \neq \lambda_j$  if and only if  $i \neq j$ . Namely, we can obtain the structure of the algebra  $F\mathfrak{X}$  from the multiplicities of the eigenvalues of  $B_1$ .

*Proof.* From the Chinese remainder theorem and the isomorphism  $A_i \mapsto B_i$ , it is enough that we consider the Jordan normal form decomposition of  $B_1$ . In the latter part, let  $\varphi_{B_1}(x)$  be the minimal polynomial of  $B_1$ . Since  $F\mathfrak{X}$  is generated by the single element  $B_1$ , the dimension of  $F\mathfrak{X}$  is equal to the degree  $\deg \varphi_{B_1}(x)$  of the minimal polynomial of  $B_1$ . On the other hand, we have the equation  $\dim F\mathfrak{X} = \sum_{i=1}^l s_i$ . Therefore we obtain  $\lambda_i \neq \lambda_j$  if and only if  $i \neq j$ .  $\square$

**4.1. The modular adjacency algebras of Grassmann graphs.** In this subsection, we consider the modular adjacency algebras of  $P$ -polynomial schemes with the  $p$ -intersection array  $\{0, \dots, 0; 1, \dots, 1\}_p$ , where the  $p$ -intersection array is the intersection array modulo  $p$ . The class of these schemes includes association schemes from Grassmann graphs  $J_{p^f}(v, d)$ ; dual polar graphs  $B_d(p^f), C_d(p^f)$ ,  ${}^2D_{d+1}(p^f), {}^2A_{2d}(p^{2f}), {}^2A_{2d-1}(p^{2f})$ ; and half-dual polar graphs  $D_{m,m}(p^f)$  (See [2]).

In this case, we have

$$B_1 \equiv \begin{pmatrix} 0 & 1 & & & & \\ 0 & -1 & 1 & & & \\ & 0 & -1 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 0 & -1 \end{pmatrix} \pmod{p},$$

and we obtain the following theorem.

**Theorem 8.** *Under the above assumption, it follows that*

$$F\mathfrak{X} \cong F \oplus F[x]/(x^d).$$

*Proof.* It follows from the assumption that the eigenvalues of  $B_1$  are equivalent to  $0^1, -1^d$  modulo  $p$ , which proves the theorem.  $\square$

By Theorems 5, 6 and 8,  $FJ_{p^f}(v, d)$  is isomorphic to  $F \oplus F[x]/(x^d)$  and it does not depend on  $v$ . The correspondence between the basis of  $FJ_{p^f}(v, d)$  and that of  $F \oplus F[x]/(x^d)$  is  $J \mapsto (1, 0)$ ,  $A_0 - J \mapsto (0, 1)$  and  $A_2 \mapsto (0, x)$ .

#### ACKNOWLEDGEMENT

The authors would like to thank Akihide Hanaki and the referees for their valuable suggestions and comments for improving this paper.

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